Peirce's Alternative Achilles Paradox

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Peirce and Zeno

Anyone who has read much of C. S. Peirce's writings on logic will know that he held the paradoxes of Zeno in low regard. At various times, he refers to the argument of Zeno against motion as "contemptible,"[1] "ridiculous,"[2] or a "wretched little catch."[3] He attributes William James' susceptibility to the Achilles paradox to "his almost unexampled incapacity for mathematical thought, combined with intense hatred for logic."[4]

In the fragment "Achilles and the Tortoise,"[5] Peirce makes it clear that his disdain is aimed not at Zeno, but at those who, although they should know better, continue to be taken in by his arguments:

> If he really conducted his attack on motion so feebly as he is represented to have done, he is to be forgiven. But that the world should continue to this day to admire this wretched little catch, which does not even turn upon any particularity of continuity, but is only a faint rudimentary likeness to an argument directed against an endless series, is less pardonable.[6]

Moreover, he notes that what we read of Zeno's arguments in Aristotle and Simplicius, who were presenting the paradoxes only in order to refute them, may not faithfully reproduce Zeno's original logic. In the guise of a dream, Peirce tells us that Zeno once

> expounded to me those four arguments; he showed me what they really had been, and why just four were needed. Very, very different from the stuff which figures for them in Simplicius. He recognized now that they were wrong, though not shallowly wrong; and he was not a little proud of having rejected the testimony of sense in his loyalty to reason.[7]

Peirce will not reveal "what they really had been," but he does present us with the challenge of his own modification of the Achilles argument.

Peirce's version of the Achilles

Peirce describes his modification of the Achilles as follows:

> Suppose that Achilles and the Tortoise ran a race; and suppose the tortoise was allowed one stadium of start, and crawled just one stadium per hour. Suppose that he and the hero were mathematical points moving along a straight line. Suppose that the son of Peleus, making fun of the affair, had determined to regulate his speed by his distance from the tortoise, moving always faster than that self-contained Eleatic by a number of stadia per hour equal to the cube root of the square of the distance between them in stadia.[8]

Mathematically, if u represents the position of Achilles, v the position of the Tortoise, and $x = u - v$ the distance by which Achilles trails the Tortoise, then the problem is to find a solution to the differential equation

$$\frac{dx}{dt} = x^{\frac{2}{3}}$$

which satisfies the initial condition $x(0) = -1$. That is, we are asked to find a function x of time t so the the rate of change of x with respect to t, which is the relative velocity of Achilles and the Tortoise, is equal to the distance between the runners to the two-thirds power.

Peirce sees two possible outcomes. In one,

> Achilles, so long as he is behind the tortoise, always moves (1) toward, and (2) faster than the tortoise. *Ergo*, he must overtake him.[9]

In the second,

> Achilles can only move beyond the tortoise by a motion relative to the tortoise. But his rule implies that when he coincides with the tortoise in place, he moves at precisely the same speed as the tortoise. Consequently, after he has caught up with the tortoise he has no motion relative to the tortoise. *Ergo*, he never can get ahead of him; and the race must finish, if not before Achilles reaches his competitor, then with the two neck and neck.[10]

There are, in fact, an infinite number of possible solutions to this equation, namely,

$$x(t) = \begin{cases} \dfrac{1}{27}(t-3)^3, & \text{if } 0 \le t \le 3, \\ 0, & \text{if } 3 < t < c, \\ \dfrac{1}{27}(t-c)^3, & \text{if } t \ge c, \end{cases}$$

where c may be any real number with $c \ge 3$, of which Peirce considers only the cases $c = 3$ (his first solution, in which Achilles passes the Tortoise) and the $c = \infty$ (his second solution, in which Achilles catches, but never passes, the Tortoise). However, Peirce's interest does not lie in the mathematical solution: "I ask (mind *not* what the answer to the mathematical problem is) but whether the two [solutions] contain any fallacy, or fallacies, and if so in what it or they consist."[11]

The final paragraph

At this point, Peirce appears to change the subject completely, suddenly bringing up the question of infinitesimals:

> The doctrine of limits is a pretty thing. In its best form, it rests upon the hypothesis of a system of numbers from which infinitesimals are excluded. All mathematical reasoning consists simply in tracing out the consequences of hypotheses. It is a familiar result to find the hypothesis self-contradictory. But attempts have been made to apply the method of limits to show the absurdity of the idea of infinitesimals. Is it not a strange logical procedure to deduce as as consequence of one hypothesis that a conflicting

> hypothesis is untrue? Of course, it is untrue if the first one remains true; but how is a mere hypothesis to decide that it cannot itself be set aside?[12]

To understand this shift in the text, we must remember Peirce's understanding of mathematics.

Peirce derived his definition of mathematics from that of his father:

> It was Benjamin Peirce, whose son I boast myself, that in 1870 first defined mathematics as "the science which draws necessary conclusions."[13] This was a hard saying at the time; but today, students of the philosophy of mathematics generally acknowledge its substantial correctness.[14]
>
> For all modern mathematicians agree with Plato and Aristotle that mathematics deals exclusively with hypothetical states of things, and asserts no matter of fact whatever; and further, that it is thus alone that the necessity of its conclusions is to be explained. This is the true essence of mathematics; and my father's definition is in so far correct that it is impossible to reason necessarily concerning anything else than a pure hypothesis. Of course, I do not mean that if such pure hypothesis happened to be true of an actual state of things, the reasoning would thereby cease to be necessary. Only, it never would be known apodictically to be true of an actual state of things.[15]

As one consequence, Peirce held that the structure of the continuum was not a question of mathematics itself, but rather a question of the choice of axioms from which one begins mathematical reasoning. In particular, Peirce held that Dedekind's construction of the continuum, although accepted by most mathematicians, "is beyond the jurisdiction of Pure Mathematics, which deals exclusively with the consequences deducible from hypotheses arbitrarily posited."[16]

Against this background, Peirce concerns himself with two approaches to the continuum: the first, based on the work of Dedekind on constructing the real numbers from the rationals and using the theory of limits, developed by Cauchy, Weierstrass, and others, to obtain the results of calculus upon the hypothesis that the continuum consists of no more than the real numbers, and the second, with roots going back to Leibniz and earlier, which supposes that, beyond the real numbers, the continuum also contains infinitely small and infinitely large elements. In the latter, the introduction of limits is not necessary in order to develop the results of calculus. Peirce contends that a mathematician is free to base his work on either assumption, in contrast to those who hold that only a mathematics based on a theory of limits and the continuum of Dedekind is free of logical contradiction. A prominent example of the latter type of mathematician is Simon Newcomb.

Peirce and Newcomb had tangled over Peirce's definitions of limit, doctrine of limits, and infinitesimal in the the *Century Dictionary* of 1889.[17] Here Peirce had said that a nonzero quantity i satisfying $A + i = A$, where A is any real number and by "=" we mean "measurable equality,"is an infinitesimal. Newcomb replied that if $A + i = A$, then we must conclude, from "Euclid's axiom," that $i = 0$.[18] That is, he argues that subtracting A from both sides of the equation leaves the equation $i = 0$. To Peirce the conclusion $i = 0$ is valid, but means only that we cannot measure any difference between 0 and i, not that i is actually 0. Newcomb concludes that Peirce's mathematics is out of date, that he is not aware of the developments of the theory of limits through the middle part of the nineteenth century, developments which he sees as banishing infinitesimals from mathematics. Of course, Peirce is fully aware of the mathematical theory of limits constructed by Cauchy and Weierstrass, and replies that it is Newcomb's mathematics which are in fact out of date:

> You [Newcomb] say my treatment of *limit* and *infinitesimal* are not in accord with the best mathematical thought of the day. But allow me to say that as you have not read, or at least not deeply considered the papers of G. Cantor, you are in the situation toward the subject in which twenty years ago those mathematicians were who had not read or had not studied Lobatchewsky, in reference to that subject.[19]

Peirce sees Cantor's development of a logic of infinite numbers as also providing a logic of infinitely small numbers. This is a bold statement, not accepted by Cantor himself, as pointed out by Josiah Royce.[20] In reply to Royce, Peirce contends that one may infer Cantor's implicit acceptance of infinitesimals in his discussion of the kinetic theory of gases in which he allows for a countable number of atoms in a finite amount of space. Peirce concludes from this that "if there are an infinite multitude in a finite space, the infinitesimals must be actual real distances, and not the mere mathematical conceptions, like $\sqrt{-1}$."[21]

The continuum

It appears then that this note on Zeno was not really about Zeno and the Achilles after all. Rather, Peirce wanted to make an important point about the nature of mathematics and its position within the rest of philosophy. Mathematics begins with a set of axioms and proceeds by deduction from these axioms. In his version of the Achilles, the hypotheses do not determine a unique solution. You cannot assume one solution and then argue that the others are false by showing they are inconsistent with your assumption. In short, to have a unique solution to the problem one must adopt another hypothesis.

Peirce contends that this is the situation in regard to the structure of the continuum. It is not surprising that the assumption that the continuum is identifiable with the real numbers of Dedekind's construction excludes infinitesimals, but it is a question begging exercise. Whether or not infinitesimals exist is a question of meta-mathematics: does Dedekind's construction fully capture the continuum or not? Peirce contends that it does not.

Peirce does not disagree that the theory of limits provides a basis upon which to build calculus. However, he sees this approach as a nominalistic attempt to avoid the question of infinitesimals:

> Nominalistic analyses, of which the doctrine of limits is an example, are certainly exceedingly illuminative. Nevertheless, in the most signal cases they have turned out inadequate, after all.[22]

That is, the definition of a limit provides language for talking about the infinitely small and the infinitely large without asserting anything about their reality. For the nominalist, the question of infinitesimals ends here, perhaps with an application of Occam's razor. However, for Peirce, who styles himself a realist in the mode of the scholastics, there is far more to consider.

First, Peirce holds that the existence of infinitesimals has support outside of mathematics. In particular, he argues (for example, in "The Law of Mind"[23]) that the flow of time and our consciousness of it is hard to explain without the use of infinitesimals:

> The argument which seems to me to prove, not only that there is such a conception of continuity as I contend for, but that it is

> realized in the universe, is that if it were not so, nobody could have any memory. If time, as many have thought, consists of discrete instants, all but the feeling of the present instant would be utterly non-existent.[24]

Second, his formulation of mathematics as "the science which draws necessary conclusions" leads him to accept the existence of a mathematical object as long as hypothesizing its existence does not involve a contradiction. That is, a mathematician is free to define objects as she or he deems necessary, the only constraint being that of consistency. David Hilbert had come to a similar conclusion (although for different reasons, and without the realist trappings) around the same time, precipitating a lively exchange with Gottlob Frege on the nature of mathematical objects.[25] For example, in a letter to Frege dated 29 December, 1899, Hilbert wrote:

> If the arbitrarily given axioms do not contradict one another, then they are true, and the things defined by the axioms exist.[26]

In this view infinitesimals exist as long the supposition of their existence does not imply a contradiction. Peirce thinks it "manifest that it involves no contradiction," but admits that "it is easier to make sure that a hypothesis is contradictory than that it is not so."[27] Indeed, the problem of consistency was to become a major focus of mathematical and logical thought in the 20th century. Eventually, this work would lead to Abraham Robinson's reconstruction of calculus in terms of infinitesimals,[28] justifying Peirce's insights.

Joseph Dauben has argued that "Peirce's most important reason for insisting that his infinitesimals were acceptable was their self-consistency."[29] It is certainly true that Peirce repeatedly returns to this point, yet he would not even be proposing the existence of infinitesimals if it were not for his aversion, as a scholastic realist, for nominalistic solutions and his need, as a pragmaticist, for a hypothesis to account for a theory of continuity which would explain consciousness and the flow of time.

Solutions of differential equations

It is possible that Peirce had in mind an even deeper connection between infinitesimals and his version of the Achilles, a connection which could be construed to show the inadequacy of the continuum of Dedekind.

Suppose one wishes to find a numerical approximation to the differential equation

$$\frac{dx}{dt} = f(x,t), \text{ with initial condition } x(0) = a,$$

on an interval $[0, T]$. A simple approach, known as Euler's method, is to divide $[0, T]$ into N subintervals of equal length Δt and create a sequence $\{x_i\}_{i=0}^{N}$ by setting $x_0 = a$ and

$$x_i = x_{i-1} + f(x_{i-1}, (i-1)\Delta t)\Delta t, \text{ for } i = 1, 2, \ldots, N.$$

We may then use x_i as an approximation for $x(i\Delta t)$, with the accuracy of the approximation quantifiable in terms of the step-size Δt and the function f. In words, we are creating an approximate solution to the differential equation on the interval $[0, T]$ by successively approximating the graph of the unknown function x by its tangent line.

If Δt is infinitesimal, and, consequently, N is infinite, then Euler's method provides an exact solution to the differential equation. Moreover, it has been shown[30] that, for the equation Peirce uses in the Achilles, the entire family of solutions may be generated by applying Euler's method to the equation with infinitesimal perturbations of the initial conditions. That is, starting with $x(0) = -1 + \epsilon$, where ϵ is an infinitesimal, one may recover all solutions to the equation. Hence whether or not the Tortoise ever passes Achilles, and when he does so, depends on the amount of infinitesimal perturbation ϵ.

There are no indications that Peirce, or any other mathematician of his day, was aware of this connection. Nevertheless, given Peirce's mathematical abilities, one cannot rule out entirely that he had some notion of the role of infinitesimals in differential equations of this type. Perhaps he was challenging mathematicians to see that, in order to understand why there is no contradiction in his two solutions, they need to allow infinitesimals into the mathematical universe. Indeed, with the fuller continuum provided by the non-standard reals, one sees more clearly how to generate the multitude of solutions to this equation. With this insight, one might agree more readily with Peirce that the theory of limits, although "exceedingly illuminative," is ultimately inadequate.

Notes

1 . Charles S. Peirce, *Collected Papers of Charles Sanders Peirce*, ed. C. Hartshorne and P. Weiss, Vols. 1-6 (Cambridge: Harvard University Press, 1960), 6.467.

2 . *Ibid.*, 6.177.

3 . Charles S. Peirce, *The New Elements of Mathematics*, ed. Carolyn Eisele, Vol. 3 (The Hague: Mouton Publishers, 1976), 119.

4 . *Collected Papers*, 6.182.

5 . *New Elements of Mathematics*, 118-120.

6 . *Ibid.*, 119.

7 . *Ibid.*

8 . *Ibid.*

9 . *Ibid.*, 119-120.

10 . *Ibid.*, 120.

11 . *Ibid.*

12 . *Ibid.*

13 . Benjamin Peirce, "Linear Associative Algebra," *American Journal of Mathematics*, Vol. 4, Issue 1/4 (1881), 97.

14 . *Collected Papers*, 4.229.

15 . *Ibid.*, 4.232.

16 . *Ibid.*, 6.183.

17 . *The Century Dictionary and Cyclopedia*, Vol I-X (New York: The Century Co., 1889).

18 . Carolyn Eisele, "The Charles S. Peirce-Simon Newcomb Correspondence," *Proceedings of the American Philosophical Society*, vol. 101, no. 5 (October, 1957), 419.

19 . *Ibid.*, 417.

20 . Royce, *The World and the Individual: Gifford Lectures delivered before the University of Aberdeen. 1st Series: The four historical conceptions of*

being (New York: Macmillan, 1899), 562.

21 . C. S. Peirce, "Discussion and Correspondence: Infinitesimals," *Science*, New Series, vol. 11, no. 272 (March 16, 1900), 430-433.

22 . *New Elements of Mathematics*, 124.

23 . Charles S. Peirce, *The Essential Peirce: Selected Philosophical Writings*, Vol. 1 (Bloomington: Indiana University Press, 1992), 312-333.

24 . *Collected Papers*, 4.641.

25 . *Gottlob Frege: Philosophical and Mathematical Correspondence*, ed. G. Gabriel, H. Hermes, F. Kambartel, C. Thiel, and A. Veraart, trans. Hans Kaal (Chicago: University of Chicago Press, 1980), 31-52.

26 . *Ibid.*, 42.

27 . *New Elements of Mathematics*, 122.

28 . Abraham Robinson, *Non-Standard Analysis* (Amsterdam: North-Holland, 1966).

29 . Dauben, Joseph, "Peirce's Place in Mathematics," *Historia Mathematica*, Vol. 9 (1982), 319.

30 . Birkeland and Normann, "A Non-Standard Treatment of the Equation $y' = f(y,t)$," *Preprint Series - Matematisk institutt, Universitetet i Oslo*, no. 17 (December 4, 1980), 6. Also see Albeverio, Høegh-Krohn, Fenstad, and Lindstrøm, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics* (New York: Academic Press, 1986), 32.